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LETTER TO THE EDITOR

The ground-state solutions of the conformally covariant spin-2 wave equation

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Abstract. The most general ground-state solutions of the second-order conformally covariant spin-2 wave equations are obtained. The spin-2 field is represented by the symmetric, rank-two tensor of the conformal group. We find that none of the ground-state solutions belong to a unitary irreducible representation of the conformal group.

1. Introduction

In the previous letter by Barut and Xu (1982) it was pointed out that the massless spin-2 Fierz–Pauli wave equation (or the linearised Einstein equation) is not conformally covariant. This result is contrary to what we used to expect for massless theories. It is surprising that, by adopting a different set of coefficients, this second-order wave equation can be made conformally covariant again. In the present letter we shall give the general ground-state solutions of this conformally covariant wave equation and we show that none of the ground state solutions belong to a unitary irreducible representation of the conformal group (Mack 1977).

We establish our proof by identifying the previous conformally covariant spin-2 wave equation written in four-dimensional Minkowski space with the manifestly conformally covariant wave equation written in Dirac's six-cone formalism (Dirac 1936). The most general three families of ground-state solutions are then obtained in six-dimensional space. They are characterised by the sets of three quantum numbers (E_0, j_1, j_2). E_0 is the lowest eigenvalue of the conformal energy operator and (j_1, j_2) characterises the finite-dimensional irreducible representation of the compact subgroup $SU(2)\times SU(2)$. In order to carry a unitary irreducible representation of the conformal group, the set of quantum numbers (E_0, j_1, j_2) of the ground-state solution of the conformally covariant wave equation should at least satisfy either of two inequalities (see below). However, we find that none of three ground-state solutions can satisfy this criterion.

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2. Manifestly conformally covariant wave equation

The conformal group of Minkowski space can be realised as the $SO(4,2)/Z_2$ group on four-dimensional conformal space

$$y^2 = y_1^2 + y_2^2 + y_3^2 - y_0^2 + y_5^2 - y_6^2 = 0, \qquad y_a \simeq \lambda y_a, \qquad \lambda \neq 0, \tag{1}$$

embedded in a six-dimensional pseudo-Euclidean space with diagonal metric tensor $g_{ab} = +, +, +, -, +, -$, $a, b = 1, 2, 3, 0, 5, 6$. The 15 generators J_{ab} of $SO(4,2)$ satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \mathrm{i}(g_{ac}J_{bd} + g_{bd}J_{ac} - g_{bc}J_{ad} - g_{ad}J_{bc}). \tag{2}$$

To employ the classification of representations of the conformal group with respect to the maximal compact subgroup $U(1)\times SU(2)\times SU(2)$, we will calculate the ground states with respect to the conformal energy J_{06}. The energy lowering operators are then $J_{0d} + \mathrm{i}J_{6d}$, $d = 1, 2, 3, 5$. The ground states denoted collectively by $|0\rangle$ are defined by

$$J_{06}|0\rangle = E_0|0\rangle, \qquad (J_{0d} + \mathrm{i}J_{6d})|0\rangle = 0. \tag{3), (4}$$

As usual we use a rank-two, symmetric tensor $H_{ab}(y)$ for the field of a spin-2 particle. The generator J_{ab} of $SO(4,2)$ will take the standard form. However, it is a little out of standard fashion but completely equivalent and more convenient for our purpose here to introduce a different form for J_{ab},

$$J_{ab} = -\mathrm{i}[y_a(\partial/\partial y_b) - y_b(\partial/\partial y_a)] - \mathrm{i}[z_a(\partial/\partial z_b) - z_b(\partial/\partial z_a)], \qquad a, b = 1, 2, 3, 0, 5, 6. \tag{5}$$

J_{ab} now acts on the index-free spin-2 field $H(y, z)$, $H(y, z) \equiv H_{ab}(y)z^a z^b$.

The second-order manifested conformally covariant wave equation is chosen to be

$$\partial^2 H_{ab}(y) = 0, \qquad \partial^2 \equiv \partial^a \partial_a, \tag{6}$$

with subsidiary conditions

$$y^a H_{ab} = 0, \qquad \partial^a H_{ab} = 0. \tag{7), (8}$$

$H_{ab}(y)$ is taken to have the degree of homogeneity $n = -1$ in y,

$$y^a \partial_a H_{bc}(y) = nH_{bc}(y). \tag{9}$$

With this choice only the wave operator ∂^2 is intrinsically defined on the conformal space (1).

The second-order conformally covariant wave equation in four-dimensional Minkowski space for a rank-two, symmetric tensor field was found previously by Barut and Xu. It was given in the form

$$\partial^2 h_{\mu\nu} - \tfrac{2}{3}\partial_\nu(\partial\cdot h)_\mu - \tfrac{2}{3}\partial_\mu(\partial\cdot h)_\nu + \tfrac{1}{3}\delta_{\mu\nu}(\partial\partial : h) = 0, \tag{10}$$

which differs from the massless spin-2 Fierz–Pauli wave equation by adopting a different set of coefficients.

To identify the above wave equation with the set of equations (6)–(8), we introduce the Minkowski space coordinates $x = (x_\mu, x_+, x_B)$, $\mu = 1, 2, 3, 0$. The pseudo-Euclidean coordinates y are then related to the Minkowski coordinates x by the standard transformation relations,

$$x_\mu = (1/x_+)y_\mu, \qquad x_+ = y_5 + y_6, \qquad x_B = y^2/x_+^2.$$

The spin-2 field variable $h_{ab}(x)$ in the Minkowski space is related to the $H_{ab}(y)$ in six-dimensional pseudo-Euclidean space by

$$h_{ab}(x) = (\partial y^c/\partial x^a)(\partial y^d/\partial x^b)H_{cd}(y),$$
$$a, b = 1, 2, 3, 0, +, B, \qquad c, d = 1, 2, 3, 0, 5, 6. \tag{11}$$

In terms of x-coordinates the $\mu\nu$ components of the wave equation (6) become

$$\partial_\lambda \partial^\lambda (h_{\mu\nu} + h_{\nu B}x_\mu + h_{\mu B}x_\nu) = 0. \tag{12}$$

The $\mu 5$ and $\mu 6$ components of the wave equation (6) together imply

$$\partial_\lambda \partial^\lambda [2h_{\mu B} + 4h_{BB}x_\mu] = 0. \tag{13}$$

Two subsidiary conditions equations (7) and (8) imply

$$\begin{aligned} &h_{+\mu} = h_{++} = h_{+B} = 0, \qquad h^\mu_\mu = 0, \\ &h_{\mu B} = -[1/2(n+4)](\partial\cdot h)_\mu, \qquad h_{BB} = [1/4(n+4)(n+3)](\partial\partial : h). \end{aligned} \tag{14}$$

Substituting (14) into (12) and (13), with $n = -1$, and combining the two equations, leads exactly to the conformally covariant wave equation (10).

From now on we shall deal only with wave equation and subsidiary conditions, equations (6)–(8), instead of the more intricate wave equation (10). It is much easier to obtain the most general ground-state solutions in the six-dimensional pseudo-Euclidean space than in the four-dimensional Minkowski space.

3. Non-unitarity of the ground-state solutions

All the unitary irreducible representations (UIR) with positive energy of the conformal group SU(2, 2) have been classified by Mack. According to Mack, all the physically interesting UIR (with positive energy) should satisfy either of the following inequalities:

(i) $E_0 \geqslant j_1 + j_2 + 2$ if $j_1 j_2 \neq 0$,

(ii) $E_0 \geqslant j_1 + j_2 + 1$ if $j_1 j_2 = 0$,

where E_0 is the lowest eigenvalue of the energy operator J_{06}. We want to use this criterion to study our ground-state solutions of the conformally covariant wave equation.

As we mentioned before it is much easier to find the most general ground-state solutions by using the wave equations written in six-dimensional space, equations (6)–(8). However, we can further simplify our effort by first looking into the set of linear equations for ground states, equations (3) and (4). For $n = -1$ there are only three families of ground-state solutions ($e, d = 1, 2, 3, 5$; $y_+ = y_0 + iy_6$):

(i) $E_0 = +1$, $(j_1, j_2) = (1, 1)$,

$$y_+^{-1}(y_+^{-2}y_d y_e z_+ z_+ - y_+^{-1}y_e z_d z_+ - y_+^{-1}y_d z_e z_+ + z_d z_e);$$

(ii) $E_0 = 0$, $(j_1, j_2) = (1/2, 1/2)$,

$$y_+^{-1}(y_+^{-1}y_d z_+ z_+ - z_+ z_d);$$

(iii) $E_0 = -1$, $(j_1, j_2) = (0, 0)$,

$$y_+^{-1}z_+ z_+.$$

None of these ground states belongs to a unitary representation of SU(2, 2). In addition none has the same Casimir eigenvalues as $D(3, 2, 0)$ or $D(3, 0, 2)$. The latter are the unique unitary extensions of the massless helicity ±2 representations of the Poincaré group. So gravitons cannot appear in an indecomposable representation either. We conclude that a symmetric 2-tensor with $n = -1$ is unable to describe gravitons in conformal space. Only the first family of ground states also satisfies the wave equation and subsidiary conditions equations (6)–(8). So the conformal spin-2 field equation (10) has as solution space a non-unitary spin-2 representation $D(1, 1, 1)$, and no physical gravitons.

In order to find conformal field equations which carry gravitons, we are forced to extend our search to fields of higher rank.

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